Approach to the chaotic synchronized state of some driving methods

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The aim of the present work is to analyze the approach to the asymptotic synchronized state of two different methods of coupling nonlinear systems exhibiting chaotic behavior through driving, namely, the Pecora-Carroll method [Phys. Rev. Lett. **64**, 821 (1990)] and a recently introduced modification to this method [Phys. Rev. E **52**, R2145 (1995)]. The stability of the different connections can be discussed in terms of the corresponding transverse Lyapunov exponents. In addition, further information can be obtained by studying the local stability of the dynamics of the differences between the two coupled systems. The main behaviors found for these systems are illustrated by using suitable models. [S1063-651X(97)03801-4]

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I. INTRODUCTION

Many situations can be represented in terms of coupled nonlinear oscillators [1,2] in scientific areas such as optics, communications, condensed-matter physics, chemical reactions, and biology. Studies on these systems show that a collection of oscillators may become synchronized to the same frequency and phase if the strength of the coupling exceeds a certain threshold value. On the other hand, many phenomena occurring in nonequilibrium systems exhibit a dependence on spatial degrees of freedom. These systems are formally modeled through partial differential equations, which yield an infinite-dimensional problem, e.g., when analyzed in terms of a Fourier series. As this problem is quite intractable, a common approximation has been to study a discrete collection of coupled systems. The study of the conditions under which these periodic oscillators synchronize has aroused much interest in recent years [3], while more general systems also have been considered [4].

At first sight it might seem that chaotic systems should not be good candidates for this synchronization behavior because one of the hallmarks of deterministic chaos is the dependence of this kind of system on the initial conditions. This property implies that two trajectories that differ by a tiny amount will become completely uncorrelated after a certain amount of time has passed. Thus the theoretical [5] and later practical [6,7] demonstration of synchronization in chaotic systems, if coupled in an appropriate way, might appear to be contradictory. Synchronized behavior implies that the dynamics of the coupled system takes place on an invariant submanifold of phase space called the synchronization manifold. The stability condition for this behavior to happen is that perturbations transverse to the invariant manifold must die off exponentially.

One of the possible practical uses of chaotic synchronization is in the field of secure communications [8–10]. The idea is to mask the information-bearing signal to be transmitted with a chaotic signal that exhibits broadband features. This is an alternative to more classical noise-masking methods, in which one uses a purely stochastic signal to mask the information to be transmitted. One should note, however, that some recent studies [11] have shown that more sophisticated masking techniques should be used because otherwise the transmitted signal would be easily unmasked by using the deterministic properties of a chaotic signal.

There exist several possibilities for linking chaotic systems so that they become synchronized. Thus one can connect them by using linear coupling, as initially suggested by Fujisaka and Yamada [5] and implemented in practice by a number of authors, using both mutual and unidirectional coupling [12–14]. In this case it is guaranteed that if one couples all the variables there is some threshold value for the coupling such that one gets synchronized behavior [13] (although the practical determination of this threshold may be subtle [15]). Another possibility is to have a drive-response couple of systems so that a signal from the drive is introduced into the response [6]. To use such a technique in the field of secure communications one needs to design a chaotic filter, which can be obtained by replacing the response system by a cascade of two chaotic systems [8]. These two cascaded response systems can be designed in some circumstances in a compact way, as shown in Refs. [9,16]. For a partial list of other experimental settings and theoretical studies in which chaotic synchronization has been studied see Refs. [17–24]; synchronization has also shown its usefulness in the field of chaos control [25].

Another context in which chaotic synchronization may be interesting is related to the fact that coupled systems have a higher dimensionality than the corresponding isolated chaotic systems. Some behaviors have been already found in the case of two coupled systems, such as the existence of riddled basins [26] and on-off intermittency [27]. These behaviors are associated with the so-called blowout bifurcations [28], in which a change in the stability of perturbations transverse to the invariant manifold makes the system *blow out* from the low-dimensional invariant manifold to the whole phase space. In the first case one also needs some symmetry to be present in the system such that two invariant manifolds coexist. It might happen that other behaviors are found for arrangements with more chaotic systems. Another possible use of coupled systems is in the synthesis of new chaotic behaviors from known low-dimensional ones [29].

In recent work [30] we have suggested a method that allows one to design chaotic filters that achieve this result in a single connection in a more systematic fashion, namely, by injecting the driving signal at a precise place of the response evolution equations. This method can be useful in the design of complex networks composed out of chaotic units [31], which may be interconnected in different ways, including the situation in which one considers neurons [32], while its usefulness in an experimental setting of Chua's circuits has been recently shown [33]. If one can get a chaotic filter by means of just one of such connections, a network of nonlinear systems might act as a more powerful information processing device.

The aim of the present contribution is to perform a detailed study of the synchronization behavior of the Pecora-Carroll method [6] and of the modification of this method introduced in Ref. [30]. The synchronization condition is associated with the asymptotic stability of the synchronized state. Instead, here we wish to characterize the way in which one approaches this synchronized state and the different possible behaviors that may arise from two coupled systems, illustrating them with appropriate examples.

The basic tool for these studies will be a linear stability analysis of the difference dynamics between the drive and response systems. As the corresponding matrix contains, in general, a few nonconstant coefficients that are themselves chaotic, i.e., rapidly varying signals, an approximation will be considered. This will consist of replacing the instantaneous time-dependent coefficients, which are the system variables, of the corresponding linearized problem by their mean values. This approximation will fail in some cases, but in some other cases it will allow one to obtain further useful information about the local behavior of the approach to the synchronized state.

The plan of the present paper is as follows. In Sec. II the synchronization methods to be analyzed are discussed, together with a detailed study of their stability at a local level. Then, Sec. III considers the study of the generic behaviors followed by systems connected by using both the Pecora-Carroll method and the method introduced in Ref. [30] in terms of the corresponding stability analysis. Finally, Sec. IV gathers the main conclusions stemming from the present work.

II. METHOD

Now we shall analyze two different connection methods that are based on the unidirectional injection of a signal produced by the first system (drive) into the second system (response). In the case of the Pecora-Carroll (PC) method [6], the driving is done in such a way that the driving signal is common between the drive and response systems, i.e., the dynamical evolution of this signal is suppressed in the response. On the other hand, the modified method introduced in Ref. [30] differs in that the driving signal is not common between the two systems, entering, instead, at a particular place of the response evolution equations (see Sec. III for some practical examples).

Synchronization implies that the dynamics of the compound system collapses onto an invariant manifold in the global phase space of the coupled systems called the synchronization manifold. The dimension of this manifold is that of the response system minus the dimension of the driving signal if one deals with the PC method [6], while in the case of the modified method [30] this dimension is identical to that of both the drive and response systems. In the present study we shall consider just the case that the synchronization manifold is obtained from the *trivial* synchronization condition $\mathbf{x}'=\mathbf{x}$, with $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{x}}'=\mathbf{f}(\mathbf{x}',\mathbf{x})$ the drive and response dynamical systems, respectively, although it is possible to define synchronization in a more general sense [34]. Notice also that we are considering the case of homogeneous driving, i.e., the case in which the two dynamical systems are identical and are defined by the same parameters, although it is also possible to consider the case in which the parameters are different (see, e.g., [6,35,36]).

The stability condition for synchronization can be studied by considering the fate of perturbations that bring the system outside the synchronization manifold. At a local level this can be done by studying the time evolution of small perturbations that are transverse to the synchronization manifold, i.e., by performing a linear stability analysis of these perturbations. This analysis will yield a linear equation of the form

$$\dot{\mathbf{x}}'-\dot{\mathbf{x}}=\dot{\delta\mathbf{x}}=\mathbf{Z}\delta\mathbf{x}+O((\delta\mathbf{x})^2), \tag{1}$$

where the relevant information about the synchronization behavior of the system is contained in the matrix $\mathbf{Z}$.

Now we shall discuss the form of the evolution matrix $\mathbf{Z}$ for transverse perturbations both in the case of the original PC method [6] and in the case of the modified method introduced in Ref. [30]. First of all, it is useful to notice that in the case that the two systems are not connected, i.e., in the case that the response evolves independently, $\mathbf{Z}$ is identical to the Jacobian of the chaotic flow: $\mathbf{Z}_u=\partial\mathbf{f}/\partial\mathbf{x}$. The structure of the $\mathbf{Z}$ matrix differs in the original PC method and in the modified case. In this work we shall consider three-dimensional dissipative dynamical systems that are driven by one-dimensional signals. This implies that the dimension of the synchronization manifold will be 2 in the case of the PC method and 3 in the case of the modified method.

In the case of the PC method $\mathbf{Z}$ is obtained by deleting one row and one column from the Jacobian of the flow $\partial\mathbf{f}/\partial\mathbf{x}$. The deleted row corresponds to derivatives of the component of the flow that corresponds to the driving signal, while the deleted column corresponds to derivatives with respect to the driving signal, e.g., if the driving signal is y, then one would delete the second row and the second column. Instead, in the case of the modified method [30] $\mathbf{Z}$ is obtained by setting to zero those entries in $\partial\mathbf{f}/\partial\mathbf{x}$ that correspond to the term in the evolution equations in which the driving signal is introduced; see below.

Solving the eigenvalue problem in Eq. (1) allows one to write the following expression for the solutions [37]:

$$\delta\mathbf{x}(t)\approx\delta\mathbf{x}(0)\exp(t\mathbf{Z})=\delta\mathbf{x}(\mathbf{0})\mathbf{T}\exp(t\mathbf{\Lambda})\mathbf{T}^{-1}, \tag{2}$$

provided that the coefficients are constant, where $\mathbf{\Lambda}$ is a diagonal eigenvalue matrix in the case that $\mathbf{Z}$ is symmetric and the eigenvalues are real and distinct, and this allows a simpler determination of $\exp(t\mathbf{Z})$. Otherwise, if some eigenvalue is complex or repeated $\mathbf{\Lambda}$ will be factorized in Jordan boxes. $\mathbf{T}$ is a matrix that puts $\mathbf{Z}$ in this diagonal or more complex form and has the eigenvectors (or *generalized* eigenvectors) of $\mathbf{Z}$ as columns.

However, the most usual situation will be that the matrix $\mathbf{Z}$, obtained by deleting one row and one column from the Jacobian or setting to zero one of the elements of the latter, will have some time-dependent coefficient that arises from the fact that $\mathbf{f}$ is nonlinear. In addition, this term will have a very complicated form as it represents a chaotic signal, and one can have access to the knowledge of the asymptotic behavior of the system only by obtaining the Lyapunov spectra for variations transverse to the invariant manifold, which is an ergodic property of the system [38]. This allows one to write the following expression, valid if one wishes to determine the fate of small transverse perturbations in an average sense:

$$\delta\mathbf{x}(t)\approx\delta\mathbf{x}(0)\exp(t\mathbf{\Lambda}'), \tag{3}$$

where $\mathbf{\Lambda}'$ is a diagonal matrix containing the Lyapunov exponents, which in the present work have been obtained by employing the algorithm introduced by Wolf *et al.* [39]. Expressed in these terms, the task of determining stable connections from the viewpoint of synchronization amounts to finding submatrices, in the case of the PC method, or setting to zero some matrix elements, in the case of the modified method, of the Jacobian such that all the transverse Lyapunov exponents are negative.

While the knowledge that one has of synchronization through Eq. (3) for systems whose linear approximation of the flow has nonconstant (chaotic) coefficients is quite limited because one has access only to the asymptotic behavior, there is a procedure, albeit approximate, of having access to more information about the system. This method consists of replacing the instantaneous values of the corresponding chaotic signals by their corresponding averaged values. Although the approximation is exact for systems in one dimension, it is not valid for higher-dimensional systems. The problem appears with pairs of complex-conjugate eigenvalues, as it is possible that a rapidly varying coefficient changes the character of the discriminant from real to imaginary or vice versa, while the averaging procedure will just yield one of the two possibilities.

We shall apply these ideas first to the case of the PC method. In this case $\mathbf{Z}$ is two dimensional,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4}$$

from which one obtains the expression for the two eigenvalues,

$$\lambda_{1,2}=\frac{1}{2}[(a+d)\pm\sqrt{(a-d)^2+4\,bc}]. \tag{5}$$

The analysis of this expression indicates that one can safely ensure that the two eigenvalues are real in the case where $(a-d)^2 \gg 4\,bc$, as the expression will be linear in $(a-d)$. In other cases, one can only diagnose that the replacement by mean values will probably work well in the case where a time-dependent coefficient appears in a or d, as the fact that their contribution is squared makes it that they will probably not exhibit wild oscillations that imply a change of sign in the argument of the square root. This situation, which is associated with a null value of the argument of the square root, will be more likely in the case where b or c is time dependent.

One could think of using a more exact way of knowing whether the eigenvalues will be real or complex, namely, by determining them explicitly. This could be implemented by remembering that the Jacobian matrix is needed to obtain Lyapunov exponents. Thus one could diagonalize it at desired times, knowing explicitly the form of the eigenvalues. However, we devised the procedure outlined above based on the use of mean values as a simple strategy that could afford some useful information at a low cost. One could argue that obtaining the eigenvalues at each time is as time consuming as integrating the corresponding differential equations and characterizing the way in which the system approaches synchronization.

Regarding the modified method introduced in Ref. [30], the analysis is slightly more complex than in the case of the PC method, due to the fact that the response system, and thus also the transverse manifold, is now three dimensional, and obtaining closed formulas for the eigenvalues is more involved. However, in some cases one may perform useful relationships with the two-dimensional situations discussed for the PC method. The idea of the method is to introduce the driving signal at a precise point of the evolution equations of the response. The result is that the response system is able to regenerate the input signal and thus one is able to design a chaotic filter with a single connection, instead of using a cascade, as happens with the PC method.

We shall illustrate the method with the Lorenz system [40]. One of the possible connections consists in the evolution equations for the response

$$\dot{x}_1=\sigma(\underline{y}-x_1),\quad \dot{y}_1=Rx_1-y_1-x_1z_1,\quad \dot{z}_1=x_1y_1-bz_1, \tag{6}$$

where the convention of underlining the driving signal in the response system has been followed and the role of this signal as a time-forcing parameter is also stressed. One can also write the expression for the time evolution of the differences between the two systems

$$\dot{\mathbf{e}}=\begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{z} \end{pmatrix}=\begin{pmatrix} -\sigma & \underline{0} & 0 \\ (R-z) & -1 & -x \\ y & x & -b \end{pmatrix}\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}=\mathbf{Z}\mathbf{e}. \tag{7}$$

This expression is analogous to the error growth equation for the Lorenz system, except for the fact that in the $\mathbf{Z}$ matrix a $\underline{0}$ entry appears at the same place in which the driving signal enters (replacing σ in this case). In this example the eigenvalues are given by

$$(\sigma+\lambda)[\lambda^2+(1+b)\lambda+x^2+b]=0, \tag{8}$$

which has the solutions

$$\lambda_1 = -\sigma, \quad \lambda_{2,3} = \frac{1}{2}\left[-(b+1) \pm \sqrt{(b+1)^2 - 4(x^2+b)}\right]. \tag{9}$$

This particular example clearly illustrates the fact that the analytical solution cannot be found in general due to the presence of the variables x, y, or z in the expressions for the eigenvalues. Thus one needs to calculate the averaged counterpart of the eigenvalues by introducing the corresponding transverse Lyapunov exponents (TLEs). The numerical values for the TLEs of this system are $(-1.80,-1.87,-10.00)$, implying that the connection is stable from the point of view of synchronization. If one approximates the eigenvalues $\lambda_{2,3}$ in Eq. (9) by using the averaged value $\langle x^2\rangle \approx 60$, the result is $\lambda_{2,3} = -(b+1)/2 = -11/6 = -1.833$, which agrees quite well with the numerically determined values. The fact that this agreement is not perfect implies that $\langle x^2\rangle$ exhibits some local deviations that make positive the argument of the square root in Eq. (9).

III. RESULTS

The aim of the present section is to discuss the different possible behaviors that have been uncovered regarding the approach to the synchronized state of coupled chaotic systems. Most of these behaviors are found to be quite generic, in the sense that they appeared in a quite systematic search in a number of models exhibiting chaos for both the Pecora-Carroll synchronization method and the modification discussed in Sec. II. As already announced, the analysis is facilitated in some cases because the linearized evolution matrix is only a function of constant coefficients and thus one does not need to perform the asymptotic analysis implicit in the evaluation of the Lyapunov exponents. However, this is not true in many of the most interesting cases, although it would be very useful to get the kind of local information provided by the eigenvalues and eigenvectors of the linearized matrix, while giving results that are consistent with the asymptotic Lyapunov analysis [for the PC method this is the case in which this nonconstant coefficient is a or d in Eq. (4)].

In other cases where the averaging procedure is not correct, one has to take advantage only of the asymptotic analysis. As usual, from the information that one obtains, the imaginary part informs about whether the approximation to the synchronized state will be steady or oscillatory, while the real part, depending on whether it is positive, zero, or negative, allows one to define the following behaviors: synchronization, marginal synchronization, and nonsynchronization, respectively, permitting this to classify the observed behaviors. Except when stated otherwise, the examples presented in this section have been obtained by application of the modified method introduced in Ref. [30].

A. Synchronization behavior

This group of systems is characterized by the common feature that the real part of the eigenvalues of $\mathbf{Z}$ is negative, and we shall consider several subcases depending on the imaginary part of these eigenvalues and also taking into account whether or not they depend on at least one of the variables of the system. If all the eigenvalues are real, the approach to the synchronized state will be monotonic, while if there are two complex-conjugate eigenvalues the approach to synchronization will be oscillatory. In the case of the PC method, in Table I the connections are labeled accordingly to the different behaviors to be reported in the rest of this section. Table II summarizes the information regarding the connections that are synchronizing for the modified method of Ref. [30]. In this case the same labeling has not been used, as sometimes the behaviors to be described cannot happen for the tree variables. For example, it may happen that two of the eigenvalues of $\mathbf{Z}$ are complex conjugate, but at least one of them will always be real, and the three variables cannot display the behavior described in Sec. III A 2.

TABLE I. Compendium of the different observed behaviors for Sprott [42] (represented with the corresponding letter), Lorenz [40], Rössler [45], Van der Pol–Duffing (VPD) [41], and Chua [46] models connected by means of the Pecora-Carroll method [6]. The different behaviors obtained by driving with the x, y, and z variables are defined as follows: **SM** for monotonic synchronization behavior (see Sec. III A 1); **SO** for oscillating synchronization behavior (see Sec. III A 2); **SN** for nonmonotonic and nonoscillating synchronization behavior (see Sec. III A 3); **MC** for constant shifted marginal synchronization behavior (see Sec. III B 1); **MS** for sized marginal synchronization behavior (see Sec. III B 2); **MO** for oscillatory marginal synchronization behavior (see Sec. III B 3); and, finally, **NS** for nonsynchronization behavior (see Sec. III C).

System	x PC	y PC	z PC	TLEs
B	MC	NS	MS	(0,-1), (-0.00,-1.00)
C	MC	NS	MS	(0,-1), (-0.00,-1.00)
D	NS	NS	MO	(0.00,0.00)
E	MC	NS	NS	(0.00,-1.00)
F	NS	SN	NS	(-0.17,-0.83)
G	SN	NS	NS	(-0.14,-0.86)
H	NS	SN	NS	(-0.17,-0.83)
I	NS	MC	MO	(0.00,-1.00), (0.00,0.00)
J	NS	MO	MC	(0.00,0.00), (0.00,-2.00)
K	NS	NS	SN	(-0.11,-1.06)
L	MC	MO	NS	(0.00,0.00), (0.00,-1.00)
M	MC	MO	MC	(0.00,-1.00), (0.00,0.00), (0.00,-1.00)
N	NS	MC	MO	(0.00,-2.00), (0.00,0.00)
O	NS	NS	NS	
P	MC	NS	SN	(0,-0.39), (-0.19,-0.20)
Q	NS	NS	MC	(0.00,-1.00)
R	NS	NS	NS	
S	NS	MC	SO	(0.00,-1.00), (-0.50,-0.50)
Lorenz	SN	SM	MS	(-1.80,-1.87),(-2.67,-10.0), (0.00,-11.0)
Rössler	NS	SN	NS	(0.0,-4.5)
VPD	SO	SN	NS	(-0.49,-0.51), (-0.012,-49.85)
Chua	SO	MC	NS	(-0.50,-0.50), (-0.00,-2.488)

1. Monotonic synchronization

To illustrate this case we shall consider the example of two Lorenz systems [40] connected through the PC method, so that the response is y driven, i.e., the response system is defined as,

$$\dot{x}_1 = \sigma(\underline{y} - x_1), \quad \dot{z}_1 = x\underline{y} - bz_1, \tag{10}$$

TABLE II. Compendium of observed synchronization behaviors for the modified method introduced in Ref. [30] for the same systems described in Table I. Only the connections with nonpositive transverse Lyapunov spectrum are reported here. The connections are reported explicitly by giving the altered ODE in the response system.

System	Connection	TLEs
F	$\dot{y}=-x+0.5\underline{y}$	(-0.14,-0.16,-0.70)
G	$\dot{x}=0.4\underline{x}+z$	(-0.17,-0.20,-0.63)
H	$\dot{y}=x+0.5\underline{y}$	(-0.14,-0.15,-0.71)
K	$\dot{z}=x+0.3\underline{z}$	(-0.14,-0.20,-0.82)
O	$\dot{y}=\underline{x}-z$	(-0.061,-0.065,-0.15)
Chua	$f(\underline{x})$	(-0.056,-0.056,-10.8)
Chua	$\dot{y}=\underline{x}-y+z$	(-0.50,-0.50,-2.46)
Chua	$\dot{x}=\alpha(\underline{y}-x-f(x))$	(-0.50,-50,-2.46)
Lorenz	$\sigma\underline{y}$	(-1.80,-1.87,-10.00)
Lorenz	$R\underline{x}$	(-3.951,-4.04,-5.67)
Lorenz	$-x\underline{z}$	(0.00,-2.67,-11.00)
Rössler	$a\underline{y}$	(-0.058,-0.12,-4.24)
VPD	$\dot{y}=\underline{x}-y-z$	(-46.67, -0.49, -0.51)
VPD	$-\nu\underline{y}$	(-0.52, -0.5, -47.40)

while the stability matrix has, in this case, the form

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{z} \end{pmatrix} = \begin{pmatrix} -\sigma & 0 \\ y & -b \end{pmatrix} \begin{pmatrix} \delta x \\ \delta z \end{pmatrix}, \tag{11}$$

which has two negative real eigenvalues $\lambda_1=-\sigma$ and $\lambda_2=-b$, implying synchronization (the TLEs numerically determined coincide with these values). In this case the differences between the variables must decrease in the form $\delta\mathbf{x}(t)=\delta\mathbf{x}(0)e^{-|\lambda|t}$, and this is what is actually observed (see Fig. 1). This kind of behavior is marked by SM in Table I.

2. Oscillating synchronization

An example of oscillating behavior, characterized by a pair of complex-conjugate eigenvalues, is the case of the Van der Pol–Duffing model [41], in which the response system is written in the form

$$\dot{x}_1=-\nu(x_1^3-\alpha x_1-y_1), \quad \dot{y}_1=\underline{x}-y_1-z_1, \quad \dot{z}_1=\beta y_1, \tag{12}$$

which allows one to write the **Z** matrix in the form

$$\begin{pmatrix} -\nu(3x^2-\alpha) & \nu & 0 \\ \underline{0} & -1 & -1 \\ 0 & \beta & 0 \end{pmatrix}, \tag{13}$$

from which one obtains the eigenvalues $\lambda_1=-\nu(3x^2-\alpha)$ and $\lambda_{2,3}=-0.5\pm\sqrt{1-4\beta}/2$. The latter eigenvalues are complex conjugate because the argument of the square root is always negative. The reported TLEs are $(-46.67, -0.50, -0.50)$ and an example of this behavior can be found in Fig. 2. This kind of behavior is marked with SO in Table I.

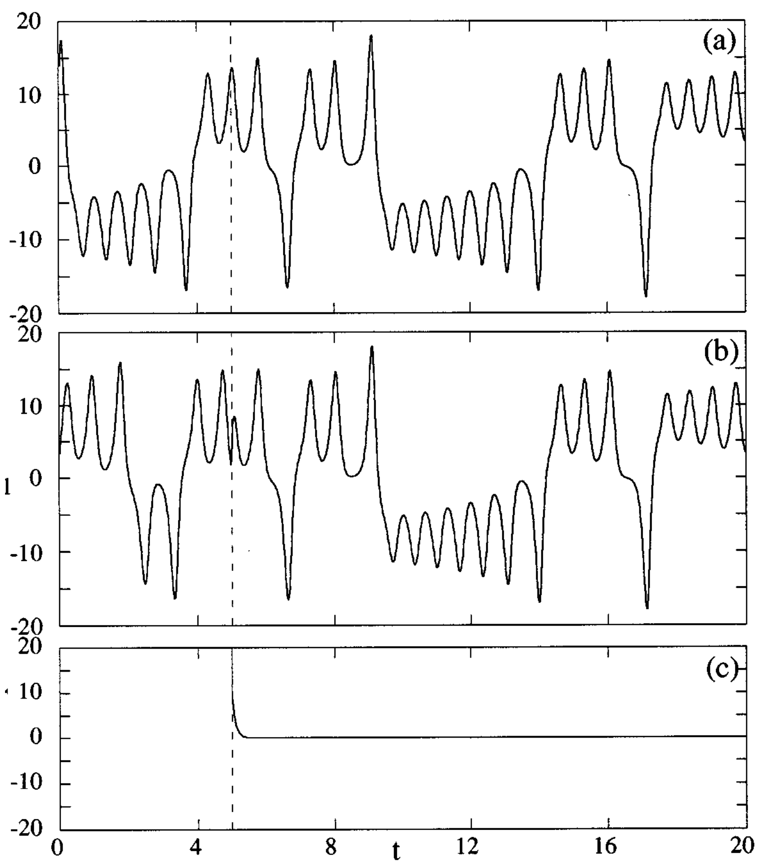

FIG. 1. Monotonic approach to chaotic synchronization in two Lorenz systems [40] coupled by using the Pecora-Carroll method [6] through y (10): (a) time evolution for x of the drive, (b) time evolution for x_1 of the response, and (c) time evolution for the difference $x-x_1$. Variable z exhibits the same type of behavior. The transverse Lyapunov spectrum is reported in Table I.

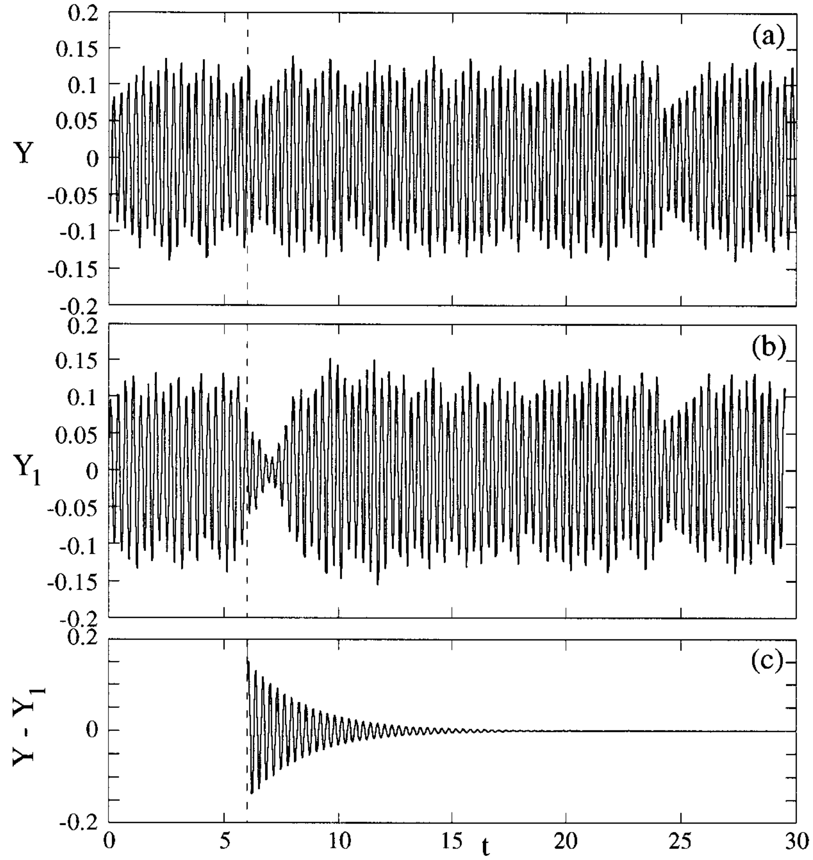

FIG. 2. Oscillatory approach to chaotic synchronization in two VPD systems [41] coupled by using the method of Ref. [30], the connection being defined in Eq. (12): (a) time evolution for y of the drive, (b) time evolution for y_1 of the response, and (c) time evolution for the difference $y-y_1$. This behavior arises from the presence of two complex-conjugate eigenvalues in a local analysis. Variable z exhibits the same type of behavior. The transverse Lyapunov spectrum is reported in Table II.

3. Nonmonotonic and nonoscillating synchronization

Systems exhibiting this behavior are characterized by an approach to synchronization that is neither monotonic nor oscillating. Although establishing a general rule is quite difficult for this case, one can say that this group is formed by those systems that have a sign-changing argument for the square-root formula (see Sec. II), and this implies that the system will exhibit an alternating oscillatory behavior and monotonic approach to synchronization.

An example of this behavior is the case of Sprott's F system [42], and considering the $\dot{y}_1=-x_1+0.5\underline{y}$ connection, the eigenvalues can be obtained by solving the algebraic equation

$$\lambda^3+\lambda^2-\lambda(2x-1)+1=0. \tag{14}$$

Considering the approximation $\langle x\rangle\approx-0.5$, it yields the eigenvalues $(-0.21\pm1.23i;-0.5863)$, while the TLEs are $(-0.14,\ -0.16,\ -0.70)$ (see also Fig. 3). This kind of behavior is marked by SN in Table I.

B. Marginal synchronization

This situation is characterized by the fact that the highest transverse Lyapunov exponent is zero, while it may happen that some of the other exponents are also zero or negative. Several qualitatively different behaviors may arise depending on whether this null TLE is zero also at a local level (if the corresponding **Z** has a zero eigenvalue) or only on the average. The behavior will also differ if there are either one or two TLEs that are null. As some of the behaviors obtained in this case are, in principle, quite unexpected, it has been the subject of a separate work [43] (see also Ref. [44] for another approach to this problem).

1. Marginal constant synchronization

As already noted, this situation is characterized by a zero transverse Lyapunov exponent, which is genuine and thus does not appear as a result of the averaging process. The behavior that one observes in such a situation is that one of the variables in the response system becomes synchronized with the drive, but with a constant separation. The connection does not have the ability to reduce the difference between drive and response systems, and the separation at the moment of the connection is preserved. Thus, in this case the synchronization manifold is not given by $\mathbf{x}=\mathbf{x}'$, but a constant appears and, moreover, this condition is not unique. In other words, what one observes in this case is $\delta x(t)=\delta x(0)$.

An example of this behavior is Sprott's N system synchronized by using the PC method through y driving. The expression for the eigenvalues is

$$\lambda(\lambda+2)=0, \tag{15}$$

in which one of the eigenvalues is identically null (see also Table I). If one solves numerically the evolution equations for both driver and response, it is observed that the response synchronizes except for a constant quantity, which is related to the separation of the two systems in phase space at the moment in which the connection starts (see Fig. 4). This kind of behavior is marked by MC in Table I.

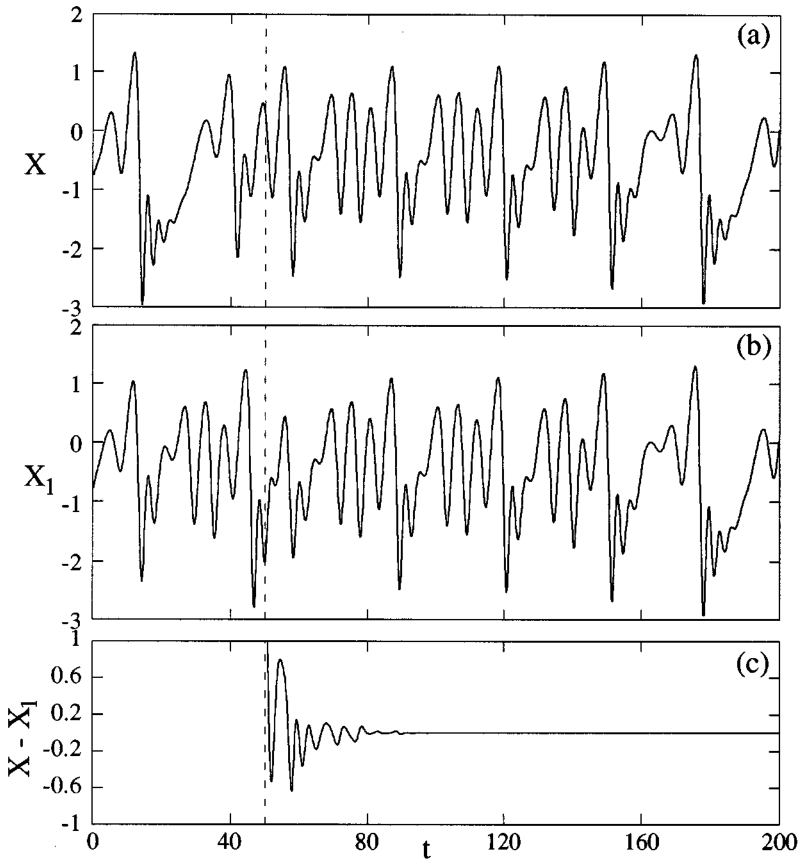

FIG. 3. Approach to chaotic synchronization behavior that is oscillatory, although not purely sinusoidal, for two Sprott F systems [45] coupled by using the method of Ref. [30], the connection being defined by $\dot{y}_1 = -x_1 + 0.5\underline{y}$: (a) time evolution for x of the drive, (b) time evolution for x_1 of the response, and (c) time evolution for the difference $x - x_1$. Variables y and z exhibit the same type of behavior. The transverse Lyapunov spectrum is reported in Table II.

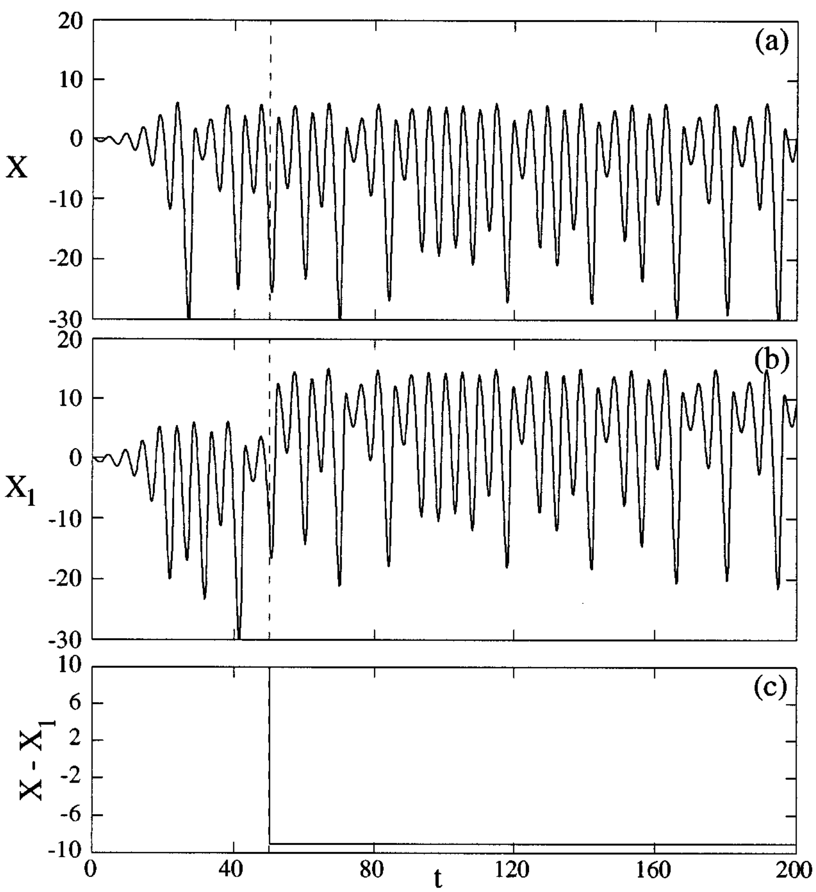

FIG. 4. Marginal constant shifted synchronization for two N Sprott systems [42] coupled by using the PC method [6] and driving with variable y: (a) time evolution for x of the drive, (b) time evolution for x_1 of the response for one initial condition yielding synchronized behavior, and (c) time evolution for the difference $x - x_1$. The transverse Lyapunov spectrum is reported in Table I.

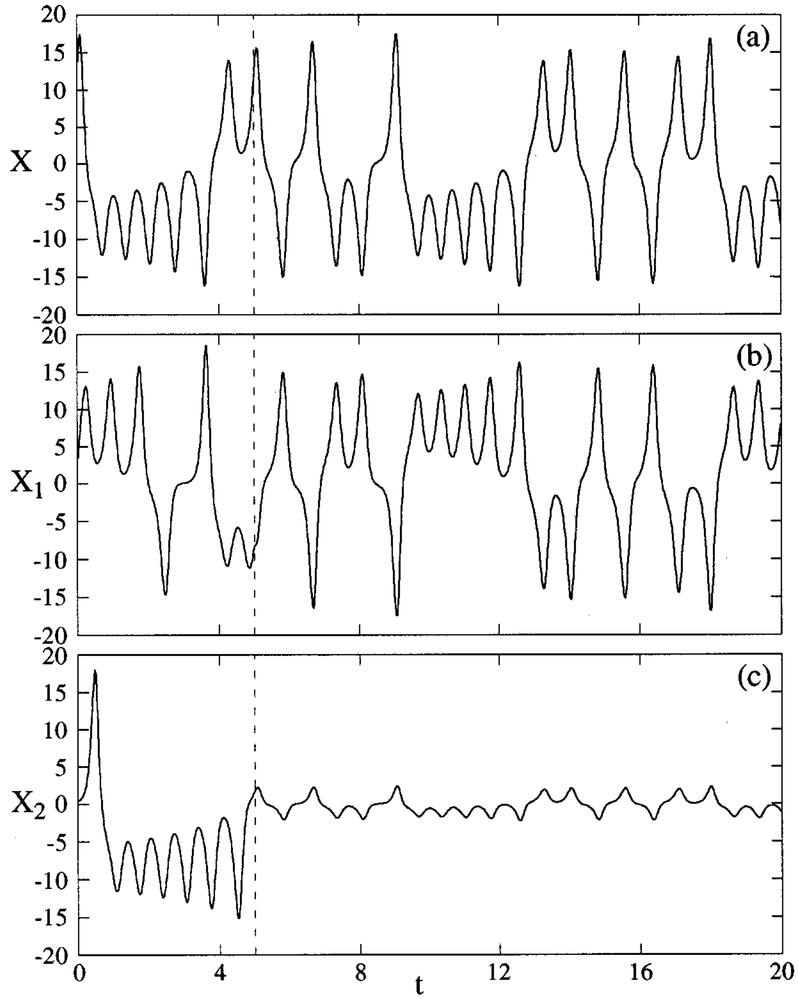

FIG. 5. Sized synchronization behavior for two Lorenz systems [40] coupled by using the method of Ref. [30], the connection being defined by $\dot{y}_1 = Rx_1 - y_1 - x_1\underline{z}$: (a) time evolution for x of the drive, (b) time evolution for x_1 of the response for one initial condition yielding synchronized behavior, and (c) time evolution for x_1 of the response for one initial condition yielding antisynchronized behavior. Variable y exhibits the same type of behavior. The transverse Lyapunov spectrum is reported in Table II.

2. Sized synchronization

The other possibility when one has a single zero transverse Lyapunov exponent is that, while the system takes this value on average, at the local level it is alternatively positive and negative. In this case the observed behavior is quite different to the situation described in Sec. III B 1 and consists in that the response exhibits the same qualitative behavior as the drive, but with different size (and sometimes with different symmetry). This different size is related to the differences in the variables referred to the moment in which the connection starts.

An example of this behavior is the Lorenz system [40], in which one drives with variable z the term $\dot{y}_1 = Rx_1 - y_1 - x_1\underline{z}$. In addition, this system has some symmetry, namely, the original system is unchanged by the transformation $(x,y,z) \to (-x,-y,z)$, and this symmetry is also present in the response system, as it is not destroyed by the connection. Thus, one has two different invariant manifolds, which are characterized by $(x_1 = x,\ y_1 = y,\ z_1 = z)$ and $(x_1 = -x,\ y_1 = -y,\ z_1 = z)$. The result is that a subset of all possible initial conditions in phase space finishes having the first type of behavior (synchronization), while the rest of the conditions finish having the second type of behavior (antisynchronization), while the sizes of the *attractors* in the drive and response systems are different. This is related to the null TLE, which avoids any reduction in the distance between the two systems (see Fig. 5 and also Ref. [43] for a more detailed study). This kind of behavior is marked by MS in Table I.

3. Marginal oscillatory synchronization

This case is characterized by the fact that **Z** has a pair of complex-conjugate eigenvalues with zero real part. This implies that the difference between the drive and response will change in an oscillatory fashion with a frequency that will depend on the imaginary part and with constant amplitude that will be related to the difference at the moment in which the connection starts. As an example we shall consider Sprott's I system [42] with connection $\dot{y}_1 = x_1 + \underline{z}$, which yields the expression for the eigenvalues

$$(\lambda + 1)(\lambda^2 + 0.2) = 0, \tag{16}$$

for which one can obtain the eigenvalues $\lambda_1 = -1$ and $\lambda_{2,3} = 0 \pm 0.45i$, implying that the two TLEs are zero, both locally and asymptotically (see Fig. 6). This kind of behavior is marked by MO in Table I.

C. Nonsynchronization behavior

When at least one of the TLEs is positive, synchronization does not occur. In certain cases, the eigenvectors corresponding to the different transverse modes do not mix the two directions (there are two independent subsystems) and it is

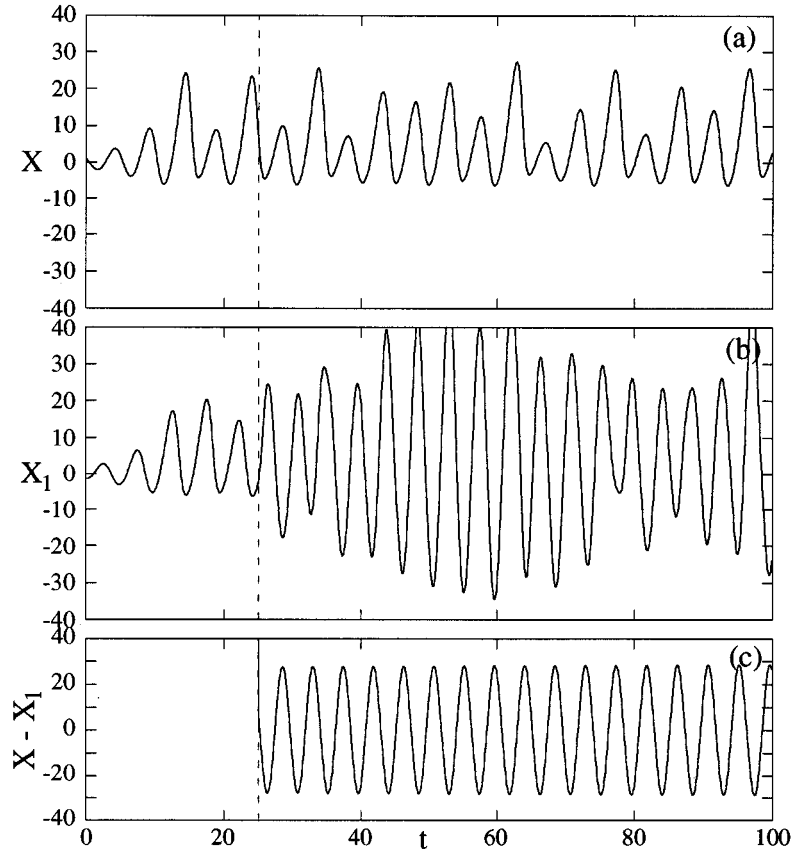

FIG. 6. Marginal oscillatory approach to synchronization for two Sprott J systems [42] coupled by using the method of Ref. [6] through y: (a) time evolution for x of the drive, (b) time evolution for x_1 of the response, and (c) time evolution for the difference $x-x_1$. Variable z exhibits the same type of behavior. The transverse Lyapunov spectrum is reported in Table I.

possible to observe that one variable synchronizes and the other one does not. In Table I connections with at least one TLE positive are indicated by NS.

1. Monotonic behavior

This type of behavior is characterized by at least one positive TLE, whose corresponding eigenvalue (obtained from a local analysis) is real. This is the case of Sprott's K system [42] with the connection $\dot{z}_1=\underline{x}+0.3z_1$. For these two cases the **Z** matrix takes the form

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{z} \end{pmatrix} = \begin{pmatrix} y & x & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0.3 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}, \tag{17}$$

which can be analyzed by decomposing it into the 2×2 matrix obtained for the PC z connection for this system, the TLEs are $(-0.11,\ -1.06)$ and a $(0.3-\lambda)$ term, yielding an unstable mode.

2. Oscillatory behavior

This type of behavior is characterized by a pair of complex-conjugate eigenvalues whose real part is positive, implying that the overall behavior will be characterized by an exponential amplification of small differences, but with a sinusoidal modulation. This is the case of the Rössler model [45] (see Fig. 7) with connection $\dot{z}_1=b+z_1(\underline{x}-c)$, which results in the eigenvalues $\lambda_1=x-c\approx -c$ and $\lambda_{2,3}=[a\pm\sqrt{a^2-4}]/2$, while the reported TLEs are $(-4.42,\ 0.10,\ 0.10)$.

3. Nonmonotonic and nonoscillatory behavior

This case occurs when one has a strongly modulating term in the real and/or the imaginary part of the eigenvalues of **Z**, resulting in a behavior with strong local variations. An example of this type of behavior is Sprott's H model [42] with the connection $\dot{y}_1=\underline{x}+0.5y_1$, and this yields the **Z** matrix

$$\begin{pmatrix} 0 & -1 & 2z \\ 0 & 0.5 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \tag{18}$$

which has the solutions $\lambda_1=0.5$ and $\lambda_{2,3}=(1\pm\sqrt{1+8z})/2$, where $\bar{z}\approx -0.4$. In practice, most of the connections studied in the present work fit in this kind of behavior.

IV. CONCLUSION

The present paper has been devoted to the detailed study of the approach to synchronization for homogeneous driving between chaotic systems by using two different procedures: first, the method introduced by Pecora and Carroll [6], and then the modification recently introduced in Ref. [30]. Synchronization is associated with a reduction of the dynamics of the system to an invariant manifold of the global phase space. Stability is analyzed by studying the fate of small

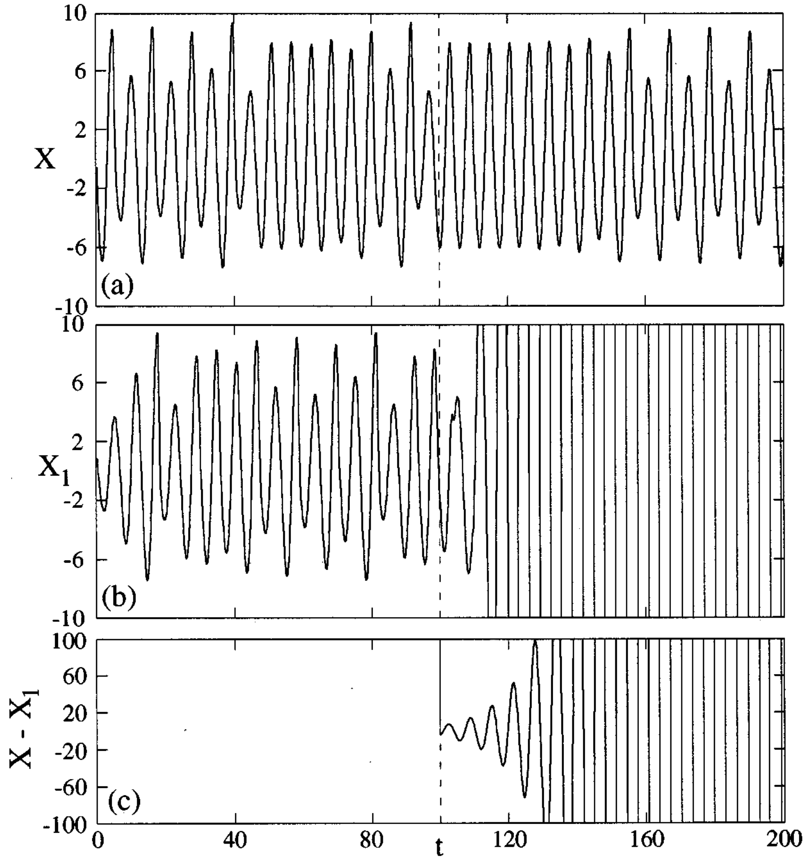

FIG. 7. Oscillatory nonsynchronizing behavior in two Rössler systems [45] coupled in the form $\dot{z}_1=b+z_1(\underline{x}-c)$: (a) time evolution for x of the drive, (b) time evolution for x_1 of the response, and (c) time evolution for the difference $x-x_1$. The transverse Lyapunov spectrum is $(-4.42,\, 0.10,\, 0.10)$.

perturbations transverse to the invariant manifold. In the first case the dimension of the transverse space is that of the original chaotic system minus that of the driving signal which is common between the two coupled systems, while in the case of the modified method the dimension of this transverse space coincides with that of an isolated chaotic system. In the examples considered in this work these dimensions are 2 and 3, respectively, implying that the analysis of the latter method is somewhat more involved.

The analysis of the linearized evolution of the differences between two coupled chaotic systems may offer a lot of useful information about the behavior of the coupled systems, from the synchronization point of view. This analysis has the difficulty that, due to the presence of nonlinear terms in the evolution equations, the corresponding linear evolution equations have nonconstant coefficients. What one usually does in this case is consider the average contribution of the linearized flow along a very long trajectory, i.e., the Lyapunov spectrum. For the purpose of synchronization one needs to determine the transverse Lyapunov exponents, which give information about the evolution of small perturbations transverse to the invariant synchronization manifold. In addition, one may try to obtain some further local information about the behavior of the coupled systems in small times, e.g., by using average values instead of the instantaneous ones, although this procedure will fail in some cases. The knowledge of the real part of the eigenvalues of this evolution matrix gives information about the ability to synchronize the coupled chaotic systems, while, instead, the imaginary part is relevant to knowing whether this approach will be monotonic or oscillatory.

Deserving special attention is the case in which the two coupled chaotic systems exhibit a marginal behavior from the synchronization point of view, this behavior happening when the highest transverse Lyapunov exponent is zero (see also [43,44]). Several possibilities may arise, depending on whether the corresponding direction is marginal at all times or only on average. In the first case the drive and response will exhibit a constant difference as a function of time. The behavior that one obtains in the second case is very interesting, because what one observes for the drive and response system is that the corresponding attractor has the same shape, its size being different. If there are more than one transverse Lyapunov exponents that are null, then the differences between the variables will vary in an oscillatory fashion.

The kind of systematic study that has been carried out in the present work leads us to wonder about how common synchronization in chaotic systems is. The first conclusion is that synchronization appears to be closely related to the degree of nonlinearity of the models that are coupled. Thus the models introduced in Ref. [42] are the simplest possible ones, from the point of view of the available nonlinearities, and one finds only six stable synchronizing connections out of a total of an average of six possible connections for each system times twenty-three systems. Instead, the more sophisticated models, such as the Lorenz [40], Van der Pol–Duffing [41], and Chua [46] systems, typically have more than one synchronizing connection each system.

As a final remark, coupling chaotic dynamical systems offers the possibility of building higher-dimensional dynamical systems. If the coupling is such that the systems synchronize, one gets an effectively lower-dimensional description of the system as the system dynamics reduces to an invariant manifold of the system. Instead, if the systems do not synchronize one would get a hyperchaotic dynamical system whose dynamical behavior can be quite rich.

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